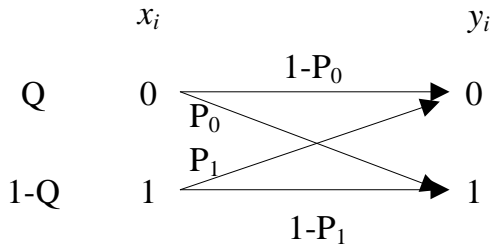


Vježbe 6

1. Odrediti kapacitet binarnog nesimetričnog kanala.

Rješenje:



$$C = \max I(X;Y),$$

$$I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X),$$

$$H(Y) = -\sum_{i=1}^N p(y_i) \log p(y_i) = -p(0) \log p(0) - p(1) \log p(1).$$

Razmotrimo $p(y)$:

$$P(Y=0) = Q(1-P_0) + (1-Q)P_1,$$

$$P(Y=1) = (1-Q)(1-P_1) + QP_0,$$

pa uvrštavanjem u formulu za entropiju dobijamo:

$$H(Y) = -[(Q(1-P_0) + (1-Q)P_1) \log(Q(1-P_0) + (1-Q)P_1) + ((1-Q)(1-P_1) + QP_0) \log((1-Q)(1-P_1) + QP_0)]$$

Posmatrajmo uslovnu entropiju:

$$\begin{aligned} H(Y|X) &= -\sum_x \sum_y p(x,y) \log p(y|x) = -\sum_x \sum_y p(x) p(y|x) \log p(y|x) = \\ &= -\sum_y (p(0)p(y|0) \log p(y|0) + p(1)p(y|1) \log p(y|1)) = \\ &= -[p(0)p(0|0) \log p(0|0) + p(0)p(1|0) \log p(1|0) + p(1)p(0|1) \log p(0|1) + p(1)p(1|1) \log p(1|1)] = \\ &= -[Q(1-P_0) \log(1-P_0) + QP_0 \log P_0 + (1-Q)P_1 \log P_1 + (1-Q)(1-P_1) \log(1-P_1)] = \\ &= -Q[(1-P_0) \log(1-P_0) + P_0 \log P_0] - (1-Q)[P_1 \log P_1 + (1-P_1) \log(1-P_1)]. \end{aligned}$$

$$\begin{aligned} I(X;Y) &= H(Y) - H(Y|X) = \\ &= -[(Q(1-P_0) + (1-Q)P_1) \log(Q(1-P_0) + (1-Q)P_1) + ((1-Q)(1-P_1) + QP_0) \log((1-Q)(1-P_1) + QP_0)] + \\ &\quad + Q[(1-P_0) \log(1-P_0) + P_0 \log P_0] + (1-Q)[P_1 \log P_1 + (1-P_1) \log(1-P_1)]. \end{aligned}$$

Imajući u vidu definiciju kapaciteta kanala, tražimo prvi izvod združene informacije:

$$*** (\log y)' = \frac{y'}{y}$$

$$\begin{aligned} \frac{dI}{dQ} &= -[(1-P_0-P_1)\log(Q(1-P_0)+(1-Q)P_1) + (Q(1-P_0)+(1-Q)P_1)\frac{(1-P_0-P_1)}{(Q(1-P_0)+(1-Q)P_1)}] - \\ &\quad - [(-1+P_1+P_0)\log((1-Q)(1-P_1)+QP_0) + ((1-Q)(1-P_1)+QP_0)\frac{(-1+P_1+P_0)}{((1-Q)(1-P_1)+QP_0)}] + \\ &\quad + [(1-P_0)\log(1-P_0) + P_0\log P_0] - [P_1\log P_1 + (1-P_1)\log(1-P_1)] = \\ &= -(1-P_0-P_1)\log(Q(1-P_0)+(1-Q)P_1) - (1-P_0-P_1) + \\ &\quad + (-1+P_1+P_0)\log((1-Q)(1-P_1)+QP_0) + (1-P_0-P_1) - H(P_0) + H(P_1) = \\ &= (1-P_0-P_1)\log\frac{(1-Q)(1-P_1)+QP_0}{Q(1-P_0)+(1-Q)P} - H(P_0) + H(P_1) = 0 \end{aligned}$$

Odnosno:

$$\log\frac{(1-Q)(1-P_1)+QP_0}{Q(1-P_0)+(1-Q)P} = \frac{H(P_0)-H(P_1)}{(1-P_0-P_1)}$$

Uvedimo da je:

$$\begin{aligned} A &= \frac{H(P_0)-H(P_1)}{(1-P_0-P_1)} \Rightarrow 2^A = \frac{(1-Q)(1-P_1)+QP_0}{Q(1-P_0)+(1-Q)P} \Rightarrow Q = \frac{1-P_1-2^A P_1}{(1-P_0-P_1)(1+2^A)} \\ \Rightarrow Q &= \frac{1-P_1-2^{\frac{H(P_0)-H(P_1)}{(1-P_0-P_1)}} P_1}{(1-P_0-P_1)(1+2^{\frac{H(P_0)-H(P_1)}{(1-P_0-P_1)}})} \end{aligned}$$

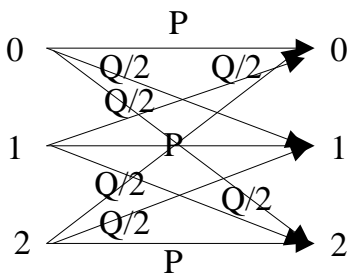
Uvrštavanjem ove vrijednosti Q u formulu za kapacitet kanala i sređivanjem izraza, dobija se njegova konačna vrijednost:

$$C = \log\left[1 + 2^{\frac{H(P_0)-H(P_1)}{1-P_0-P_1}}\right] - \frac{H(P_0)(1-P_1) - H(P_1)P_0}{1-P_0-P_1}.$$

2. Odrediti kapacitet kanala koji prenosi ternarne poruke i napisati njegovu tranzicionu matricu.

Vjerovatnoća prenosa ispravnog simbola je P dok je vjerovatnoća prenosa pogrešnog Q (po $Q/2$ da će biti prenijeta ostala dva simbola).

Rješenje:



$$C = \max I(X;Y),$$

$$I(X;Y) = H(Y) - H(Y|X).$$

Ranije je dokazano da je:

$$H(Y|X) = \sum_{x \in A} p(x)H(Y|X=x) = \\ = P(X=0)H(Y|X=0) + P(X=1)H(Y|X=1) + P(X=2)H(Y|X=2).$$

Posmatrajmo pojedinačno uslovne vjerovatnoće:

$$H(Y|X=0) = -P(Y=0|X=0)\log P(Y=0|X=0) - P(Y=1|X=0)\log P(Y=1|X=0) - \\ - P(Y=2|X=0)\log P(Y=2|X=0) = -P\log P - \frac{Q}{2}\log \frac{Q}{2} - \frac{Q}{2}\log \frac{Q}{2} = \\ = -P\log P - Q\log \frac{Q}{2}.$$

Lako ćemo uočiti da su:

$$H(Y|X=0) = H(Y|X=1) = H(Y|X=2) = -P\log P - Q\log \frac{Q}{2} = -P\log P - Q\log Q + Q\log 2 = \\ = H(P) + Q\log 2 = H(P) + Q.$$

Imajući u vidu da je: $P(X=0) + P(X=1) + P(X=2) = 1$, jer je riječ o potpunom sistemu događaja, dobijamo:

$$H(Y|X) = H(Y|X=0)(P(X=0) + P(X=1) + P(X=2)) = H(Y|X=0).$$

Dalje je:

$$I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(P) - Q,$$

odnosno:

$$C = \max I(X;Y) = \log 3 - H(P) - Q.$$

Tranziciona matrica je:

$$\begin{bmatrix} P & Q/2 & Q/2 \\ Q/2 & P & Q/2 \\ Q/2 & Q/2 & P \end{bmatrix}$$